nmNURBS

version 1.0

n-Dimensional m-Parametric NURBS Objects Formulas

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NURBS Curve Definition (Homogeneous Coordinates)

A NURBS curve could be defined in homogeneous coordinate as follows

$$C^w_{\overrightarrow{Q^w}}(v) = \sum_{j=0}^{l-1} \left[N^{\overrightarrow{H}}_{j,q}(v) \cdot Q^w_j \right]$$

where

$C^w : \mathbb{R} \rightarrow \mathbb{R}^n$	the NURBS curve function
$n \in \mathbb{N}$	number of homogeneous coordinates for each control point
$j \in \mathbb{N}$	index for the sum
$v \in \mathbb{R}$	NURBS curve parameter
$q \in \mathbb{N}$	degree of the NURBS curve
$\overrightarrow{H} \in \mathbb{R}^{n+q+1}$	vector of knots
$\overrightarrow{Q^w} \in \mathbb{R}^l \times \mathbb{R}^n$	vector of control points
$N^{\overrightarrow{H}}_{j,q} : \mathbb{R} \rightarrow \mathbb{R}$	basis function
$Q^w_j \in \mathbb{R}^n$	j-th element of the vector of control points (j-th control point)
$l \in \mathbb{N}$	number of control points

Basis Function Definition

The basis function could be defined in a recursive way as follows

$$N^{\vec{K}}_{j,q}(\nu) = \begin{cases} \dfrac{\nu - K_j}{K_{j+q} - K_j} \cdot N^{\vec{K}}_{j,q-1}(\nu) + \dfrac{K_{j+q+1} - \nu}{K_{j+q+1} - K_{j+1}} \cdot N^{\vec{K}}_{j+1,q-1}(\nu) & \text{if } q > 0 \\ \begin{cases} 1 & \text{if } K_j \le \nu < K_{j+1} \\ 0 & \text{otherwise} \end{cases} & \text{if } q = 0 \end{cases}$$

where

$N^{\vec{K}}_{j,q} : \mathbb{R} \rightarrow \mathbb{R}$	the basis function
$\vec{K}$	vector of knots
$q \in \mathbb{N}$	basis function degree
$j \in \mathbb{N}$	index for the knot vector
$\nu \in \mathbb{R}$	parameter for the basis function
$K_h \in \mathbb{R}$	h- th element of the knot vector

NURBS Object Definition (Homogeneous Coordinates)

The generalization of the NURBS curve formula (in homogeneous coordinates) is the definition of the NURBS object (in homogeneous coordinates), it is

$$R^w_{r,M^{r+1}}(\vec{v}) = \begin{cases} \sum_{j_r=0}^{l_r-1} \left[N^{\vec{K}}_{j_r,q_r}(v_r) \cdot R^w_{r-1,M^{r+1}_{j_r}}(\vec{v}) \right] & \text{if } r>0 \\ C^w_{M^1}(v_0) = \sum_{j_0=0}^{l_0-1} \left[N^{\vec{K}}_{j_0,q_0}(v_0)\, M^1_{j_0} \right] & \text{if } r=0 \end{cases}$$

where

$R^w_{r,M^{r+1}} : \mathbb{R}^m \to \mathbb{R}^n$	the NURBS object function
$n \in \mathbb{N}$	number of homogeneous coordinates for each control point
$m \in \mathbb{N}$	number of the NURBS object's parameters
$r \in \mathbb{N}$	level of recursion: each level of recursion is relative to a NURBS object's parameter
$j_r \in \mathbb{N}$	index for the sum at the r-th level of recursion
$q_r \in \mathbb{N}$	degree of the basis function relative to the r-th NURBS object's parameter
$l_r \in \mathbb{N}$	extent of the r-th dimension of the matrix of the control points
$\vec{v} \in \mathbb{R}^m$	vector of parameters for the NURBS object

$M^r \in \mathbb{R}^{l_0 \times \ldots \times l_{r-1}} \times \mathbb{R}^n$	r-dimensional matrix of control points: the extent for each dimension $d \in [0, r) \cap \mathbb{N}$ is l_d ;
more formally $M^r \in \mathbb{R}^{\times_{i=0}^{r-1} [\, l_i \,]} \times \mathbb{R}^n$	the last cartesian product by $\mathbb{R}^n$ remembers the control point structure, that is a list of n homogeneous coordinates
$M_h^r \in \mathbb{R}^{\times_{i=0, i \neq h}^{r-1} [\, l_i \,]}$, $h \in \{ [0, r) \cap \mathbb{N} \}$	an r-dimensional matrix could be defined as a vector of (r-1)-dimensional sub-matrices: in this case M_h^r stand for the h-th element (the h-th sub-matrix) of that vector

NURBS Curve Definition (Non- homogeneous Coordinates)

The classical NURBS Curve definition is

$$C_{\vec{P},\vec{W}}(v) = \frac{\sum_{j=0}^{l} \left[N_{j,q}^{\vec{K}}(v) \cdot P_j \right]}{\sum_{j=0}^{l} \left[N_{j,q}^{\vec{K}}(v) \cdot W_j \right]}$$

where

$C_{\vec{P},\vec{W}} : \mathbb{R} \rightarrow \mathbb{R}^n$	the NURBS curve function
$n \in \mathbb{N}$	number of non- homogeneous coordinates for each control point
$j \in \mathbb{N}$	index for the sum
$v \in \mathbb{R}$	NURBS curve parameter
$q \in \mathbb{N}$	degree of the NURBS curve
$l \in \mathbb{N}$	number of control points: since a weight is associated to a control point, for each control point, then the number of weights is the same as the number of control points
$\vec{H} \in \mathbb{R}^{n+q+1}$	vector of knots
$\vec{P} \in \mathbb{R}^l \times \mathbb{R}^n$	vector of control points
$P_j \in \mathbb{R}^n$	j- th element of the vector of control points (j- th control

	point)
$\overrightarrow{W} \in \mathbb{R}^l$	vector of weights
$W_j \in \mathbb{R}$	j-th element of the vector of weights
$N_{j,q}^{\overrightarrow{H}} : \mathbb{R} \to \mathbb{R}$	basis function

NURBS Curve Derivative (Non- homogeneous Coordinates)

The derivative for a NURBS Curve is

$$\frac{\partial}{\partial \nu}\left[C_{\vec{P},\vec{W}}(\nu)\right]=\frac{\partial}{\partial \nu}\left[\frac{\sum_{j=0}^{l}\left[N_{j,q}^{\vec{K}}(\nu)\cdot P_j\right]}{\sum_{j=0}^{l}\left[N_{j,q}^{\vec{K}}(\nu)\cdot W_j\right]}\right]$$

let be $F(\nu)=\sum_{j=0}^{l}\left[N_{j,q}^{\vec{K}}(\nu)\cdot P_j\right]$ and $G(\nu)=\sum_{j=0}^{l}\left[N_{j,q}^{\vec{K}}(\nu)\cdot W_j\right]$, then

$$\frac{\partial}{\partial \nu}\left[C_{\vec{P},\vec{W}}(\nu)\right]=\frac{\partial}{\partial \nu}\left[\frac{F(\nu)}{G(\nu)}\right]=\frac{\left(\frac{\partial}{\partial \nu}[F(\nu)]\right)\cdot[G(\nu)]-[G(\nu)]\cdot\left(\frac{\partial}{\partial \nu}[G(\nu)]\right)}{[G(\nu)]^2}$$

where

$$\frac{\partial}{\partial \nu}[F(\nu)]=\frac{\partial}{\partial \nu}\left[\sum_{j=0}^{l}\left[N_{j,q}^{\vec{K}}(\nu)\cdot P_j\right]\right]=\sum_{j=0}^{l}\left[\frac{\partial}{\partial \nu}\left[N_{j,q}^{\vec{K}}(\nu)\cdot P_j\right]\right]=\sum_{j=0}^{l}\left[\left(\frac{\partial}{\partial \nu}\left[N_{j,q}^{\vec{K}}(\nu)\right]\right)\cdot P_j\right]$$

and

$$\frac{\partial}{\partial \nu}[G(\nu)]=\frac{\partial}{\partial \nu}\left[\sum_{j=0}^{l}\left[N_{j,q}^{\vec{K}}(\nu)\cdot W_j\right]\right]=\sum_{j=0}^{l}\left[\frac{\partial}{\partial \nu}\left[N_{j,q}^{\vec{K}}(\nu)\cdot W_j\right]\right]=\sum_{j=0}^{l}\left[\left(\frac{\partial}{\partial \nu}\left[N_{j,q}^{\vec{K}}(\nu)\right]\right)\cdot W_j\right]$$

Basis Function Derivative

From the basis function definition, the basis function derivatives is

$$\frac{\partial}{\partial v}\left[N^{\vec{K}}_{j,q}(v)\right]=\begin{cases}\dfrac{\partial}{\partial v}\left[\dfrac{v-K_j}{K_{j+q}-K_j}\cdot N^{\vec{K}}_{j,q-1}(v)+\dfrac{K_{j+q+1}-v}{K_{j+q+1}-K_{j+1}}\cdot N^{\vec{K}}_{j+1,q-1}(v)\right] & \text{if } q>0\\ \begin{cases}\dfrac{\partial}{\partial v}[1]=0 & \text{if } K_j\leq v<K_{j+1}\\ \dfrac{\partial}{\partial v}[0]=0 & \text{otherwise}\end{cases} & \text{if } q=0\end{cases}$$

so

$$\frac{\partial}{\partial v}\left[N^{\vec{K}}_{j,q}(v)\right]=\begin{cases}\dfrac{\partial}{\partial v}\left[\dfrac{v-K_j}{K_{j+q}-K_j}\cdot N^{\vec{K}}_{j,q-1}(v)+\dfrac{K_{j+q+1}-v}{K_{j+q+1}-K_{j+1}}\cdot N^{\vec{K}}_{j+1,q-1}(v)\right] & \text{if } q>0\\ 0 & \text{if } q=0\end{cases}.$$

It is possible to explicit the v variable in the basis function's general case expression,
that is when $q>0$

$$\frac{v-K_j}{K_{j+q}-K_j}\cdot N^{\vec{K}}_{j,q-1}(v)+\frac{K_{j+q+1}-v}{K_{j+q+1}-K_{j+1}}\cdot N^{\vec{K}}_{j+1,q-1}(v)=$$

$$=\left[\left(\frac{1}{K_{j+q}-K_j}\right)\cdot v-\frac{K_j}{K_{j+q}-K_j}\right]\cdot N^{\vec{K}}_{j,q-1}(v)+\left[\frac{K_{j+q+1}}{K_{j+q+1}-K_{j+1}}-\left(\frac{1}{K_{j+q+1}-K_{j+1}}\right)\cdot v\right]\cdot N^{\vec{K}}_{j+1,q-1}(v)=$$

$$=v\cdot\left[\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}-\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right]+\left[\frac{(K_{j+q+1})\cdot\left(N^{\vec{K}}_{j+1,q-1}(v)\right)}{K_{j+q+1}-K_{j+1}}-\frac{(K_j)\cdot\left(N^{\vec{K}}_{j,q-1}(v)\right)}{K_{j+q}-K_j}\right]$$

so the basis function's derivatives in the general case $q > 0$ is

$$\frac{\partial}{\partial v}\left(v\cdot\left[\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}-\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right]+\left[\frac{(K_{j+q+1})\cdot\left(N^{\vec{K}}_{j+1,q-1}(v)\right)}{K_{j+q+1}-K_{j+1}}-\frac{(K_j)\cdot\left(N^{\vec{K}}_{j,q-1}(v)\right)}{K_{j+q}-K_j}\right]\right)=$$

$$=\frac{\partial}{\partial v}\left(v\cdot\left[\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}-\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right]\right)+\frac{\partial}{\partial v}\left(\left[\frac{(K_{j+q+1})\cdot\left(N^{\vec{K}}_{j+1,q-1}(v)\right)}{K_{j+q+1}-K_{j+1}}-\frac{(K_j)\cdot\left(N^{\vec{K}}_{j,q-1}(v)\right)}{K_{j+q}-K_j}\right]\right)=$$

$$=\left(\left[\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}-\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right]+v\cdot\left[\frac{\partial}{\partial v}\left(\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}\right)-\frac{\partial}{\partial v}\left(\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right)\right]\right)+$$

$$+\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j+1,q-1}(v)\right)\right]\cdot\left[\frac{K_{j+q+1}}{K_{j+q+1}-K_{j+1}}\right]-\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j,q-1}(v)\right)\right]\cdot\left[\frac{K_j}{K_{j+q}-K_j}\right]=$$

$$=\left[\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}-\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right]+$$

$$+\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j,q-1}(v)\right)\right]\cdot\left[\frac{v}{K_{j+q}-K_j}\right]-\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j+1,q-1}(v)\right)\right]\cdot\left[\frac{v}{K_{j+q+1}-K_{j+1}}\right]+$$

$$+\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j+1,q-1}(v)\right)\right]\cdot\left[\frac{K_{j+q+1}}{K_{j+q+1}-K\,j+1}\right]-\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j,q-1}(v)\right)\right]\cdot\left[\frac{K_j}{K_{j+q}-K\,j}\right]=$$

$$=\left[\frac{N^{\vec{K}}_{j,q-1}(v)}{K_{j+q}-K_j}-\frac{N^{\vec{K}}_{j+1,q-1}(v)}{K_{j+q+1}-K_{j+1}}\right]+$$

$$+\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j,q-1}(v)\right)\right]\cdot\left[\frac{v-K_j}{K_{j+q}-K_j}\right]+\left[\frac{\partial}{\partial v}\left(N^{\vec{K}}_{j+1,q-1}(v)\right)\right]\cdot\left[\frac{K_{j+q+1}-v}{K_{j+q+1}-K_{j+1}}\right]$$

NURBS Object Definition (Non-homogeneous Coordinates)

The generalization of the NURBS curve formula (in non-homogeneous coordinates) is the definition of the NURBS object (in non-homogeneous coordinates):

let be

$$F_{r,P^{r+1}}(\vec{v}) = \begin{cases} \sum_{j_r=0}^{l_r-1} \left[N^{\vec{K}}_{j_r,q_r}(v_r) \cdot F_{r-1,,P^{r+1}_{j_r}}(\vec{v}) \right] & \text{if } r>0 \\ C_{P^1}(v_0) = \sum_{j_0=0}^{l_0-1} \left[N^{\vec{K}}_{j_0,q_0}(v_0)\, P^1_{j_0} \right] & \text{if } r=0 \end{cases}$$

and

$$G_{r,W^{r+1}}(\vec{v}) = \begin{cases} \sum_{j_r=0}^{l_r-1} \left[N^{\vec{K}}_{j_r,q_r}(v_r) \cdot G_{r-1,,W^{r+1}_{j_r}}(\vec{v}) \right] & \text{if } r>0 \\ C_{W^1}(v_0) = \sum_{j_0=0}^{l_0-1} \left[N^{\vec{K}}_{j_0,q_0}(v_0)\, W^1_{j_0} \right] & \text{if } r=0 \end{cases}$$

then the definition of the NURBS object (in non-homogeneous coordinates) is

$$R_{r,P^{r+1},W^{r+1}}(\vec{v}) = \frac{F_{r,P^{r+1}}(\vec{v})}{G_{r,W^{r+1}}(\vec{v})}$$

where

$P^r \in \mathbb{R}^{\times_{i=0}^{r-1}[\,l_i\,]} \times \mathbb{R}^n$	r-dimensional matrix of control points: the extent for each dimension $d \in [0, r) \cap \mathbb{N}$ is l_d ; the last cartesian product by $\mathbb{R}^n$ remembers the control point structure, that is a list of n non-homogeneous coordinates
$W^r \in \mathbb{R}^{\times_{i=0}^{r-1}[\,l_i\,]}$	r-dimensional matrix of weights: the extent for each dimension $d \in [0, r) \cap \mathbb{N}$ is l_d ;

NURBS Object Derivative (Non- homogeneous Coordinates)

The NURBS object derivative formula is from the NURBS object definition, it is

$$\frac{\partial}{\partial v_h}\left[R_{r,P^{r+1},W^{r+1}}(\vec{v})\right]=\frac{\left(\frac{\partial}{\partial v_h}\left[F_{r,P^{r+1}}(\vec{v})\right]\right)\cdot\left[G_{r,W^{r+1}}(\vec{v})\right]-\left(\frac{\partial}{\partial v_h}\left[G_{r,W^{r+1}}(\vec{v})\right]\right)\cdot\left[F_{r,P^{r+1}}(\vec{v})\right]}{\left[G_{r,W^{r+1}}(\vec{v})\right]^2}$$

where

$$\frac{\partial}{\partial v_h}\left[F_{r,P^{r+1}}(\vec{v})\right]=\begin{cases}\sum_{j_r=0}^{l_r-1}\left[\left(N_{j_r,p_r}^{\vec{K}_r}(v_{j_r})\right)\cdot\left[\frac{\partial}{\partial v_h}\left(F_{r-1,P_{j_r}^{r+1}}(\vec{v})\right)\right]\right] & if\ r>0,\quad r\neq h\\ \sum_{j_r=0}^{l_r-1}\left[\left[\frac{\partial}{\partial v_h}\left(N_{j_r,p_r}^{\vec{K}_r}(v_{j_r})\right)\right]\cdot\left[F_{r-1,P_{j_r}^{r+1}}(\vec{v})\right]\right] & if\ r>0,\quad r=h\\ 0 & if\ r=0,\quad r\neq h\\ \sum_{j_0=0}^{l_0-1}\left[\left[\frac{\partial}{\partial v_h}\left(N_{j_0,p_0}^{\vec{K}_r}(v_{j_0})\right)\right]\cdot\left[P_{i_0}^1\right]\right] & if\ r=0,\quad r=h\end{cases}$$

and

$$\frac{\partial}{\partial v_h}\left[G_{r,W^{r+1}}(\vec{v})\right]=\begin{cases}\sum_{j_r=0}^{l_r-1}\left[\left(N_{j_r,q_r}^{\vec{K}_r}(v_{j_r})\right)\cdot\left[\frac{\partial}{\partial v_h}\left(G_{r-1,W_{j_r}^{r+1}}(\vec{v})\right)\right]\right] & if\ r>0,\quad r\neq h\\ \sum_{j_r=0}^{l_r-1}\left[\left[\frac{\partial}{\partial v_h}\left(N_{j_r,q_r}^{\vec{K}_r}(v_{j_r})\right)\right]\cdot\left[G_{r-1,W_{j_r}^{r+1}}(\vec{v})\right]\right] & if\ r>0,\quad r=h\\ 0 & if\ r=0,\quad r\neq h\\ \sum_{j_0=0}^{l_0-1}\left[\left[\frac{\partial}{\partial v_h}\left(N_{j_0,p_0}^{\vec{K}_r}(v_{j_r})\right)\right]\cdot\left[W_{i_0}^1\right]\right] & if\ r=0,\quad r=h\end{cases}$$